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Exact accelerating solitons in nonholonomic deformation of the KdV equation with a two-fold integrable hierarchy

Anjan Kundu

Theory Group and CAMCS, Saha Institute of Nuclear Physics Calcutta, India

E-mail: anjan.kundu@saha.ac.in

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Abstract

The recently proposed nonholonomic deformation of the KdV equation is solved through the inverse scattering method by constructing an AKNS-type Lax pair. Exact N-soliton solutions are found for the basic field and the deforming function showing an unusual accelerated (decelerated) motion. A two-fold integrable hierarchy is revealed, one with the usual higher order dispersion and the other with novel higher nonholonomic deformations.

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1. Introduction

Nonholonomic constraints on field models have been receiving increasing attention in recent years [1]. As a notable achievement such nonholonomic deformation (NHD) has been applied to an integrable system, namely to the KdV equation preserving its integrability [2]. For this system, equivalent to a sixth-order KdV equation, certain particular traveling wave solutions are found, a linear problem is formulated, and several conjectures on important issues are put forward in [2]. The main expectations for this KdV equation with nonholonomic constraint are (i) existence of an infinite set of higher conserved quantities, (ii) formulation of the Lax pair (iii) application of the inverse scattering method (ISM), (iv) N-soliton solutions for the basic and the deforming field, (v) elastic nature of soliton scattering, etc. Among these conjectures only the first one showing the existence of an integrable hierarchy with the usual higher dispersion has been proved recently [3]. Our aim here is to establish the rest with an explicit result.

In particular, we construct an AKNS-type matrix Lax pair for this NHD of the KdV equation, revealing an important connection between the time evolution of the Jost function and the NHD of the nonlinear equation. Applying subsequently the ISM we find the exact N-soliton solutions for both the field and the deforming function of the deformed KdV, which exhibit the usual property of soliton scattering, but with an unusual accelerating (or decelerating) soliton motion. Finally, we unravel a novel two-fold integrable hierarchy for this system, one with the usual higher dispersion [3] and the other yielding a new type of deformed KdV with increasingly higher order nonholonomic constraints.

2. ISM for the deformed KdV

The recently proposed [2] NHD of the KdV equation

$$u_t - u_{xxx} - 6uu_x = w_x, \tag{1}$$

$$w_{xxx} + 4uw_x + 2u_x w = 0, \tag{2}$$

can be written by eliminating the deforming function w, also as a sixth-order KdV equation for $v_x = u$:

$$\left(\partial^3_{xxx} + 4v_x\partial_x + 2v_{xx}\right)\left(v_t - v_{xxx} - 3v_x^2\right) = 0. \tag{3}$$

2.1. AKNS-type Lax pair

We intend to find the exact N-soliton solutions to the deformed KdV equations (1) and (2) for the field u and the deforming function w by the ISM, for which we construct first the AKNS-type Lax pair $U(\lambda)$, $V(\lambda)$ to formulate the linear problem $\Phi_x = U\Phi$, $\Phi_t = V\Phi$. We observe remarkably that such a Lax pair for the deformed KdV can be built up from the known pair $U_{\rm kdv}(\lambda)$, $V_{\rm kdv}(\lambda)$ for the standard KdV equation [4], by *deforming* only its time-Lax operator:

$$U(\lambda) = U_{\rm kdv}(\lambda), \qquad V(\lambda) = V_{\rm kdv}(\lambda) + V_{\rm def}(\lambda), \tag{4}$$

where the deforming operator is given as

$$V_{\rm def}(\lambda) = \tfrac{1}{2}\left(\lambda^{-1}G^{(1)} + \lambda^{-2}G^{(2)}\right) \tag{5}$$

with

$$\begin{aligned} G^{(1)} &= \mathrm{i}w\sigma^3 - w_x\sigma^+ \\ G^{(2)} &= \frac{w_x}{2}\sigma^3 + \mathrm{i}\,w\sigma^- + e\sigma^+, \qquad e_x = \mathrm{i}\,uw_x. \end{aligned} \tag{6}$$

Here, $u(x,t)$ is the KdV field, and $w(x,t)$ is the deforming function with its asymptotic $\lim_{|x|\to\infty} w = c(t)$, being an arbitrary function in time. We can check that the flatness condition $U_t - V_x + [U, V] = 0$ of the Lax pair (4) yields the deformed KdV equations (1) and (2), where the undeformed part is given by the standard pair [4]

$$U_{\rm kdv}(\lambda) = \mathrm{i}\left(\lambda\sigma^3 + U^{(0)}\right), \qquad U^{(0)} = u(x,t)\sigma^+ + \sigma^- \tag{7}$$

$$\begin{aligned} V_{\rm kdv}(\lambda) = \mathrm{i}U^{(0)}_{xx} &- 4\mathrm{i}\lambda^3\sigma^3 + 2\sigma^3\lambda\left(-U^{(0)}_x + \mathrm{i}(U^{(0)})^2\right) - 4\mathrm{i}U^{(0)}\lambda^2 \\ &+ 2\mathrm{i}(U^{(0)})^3 - \left[U^{(0)}, U^{(0)}_x\right], \end{aligned} \tag{8}$$

while the deformed part with the nonholonomic constraint is generated by the new addition (5). Therefore, we may draw an intriguing conclusion that the NHD of the KdV (1)–(2) can be linked to the deformation in the time evolution of the Jost function, which is related to $V_{\rm def}(\lambda)$ as given by (5)–(6). We see in the sequel that this fact leads to an unusual solitonic property in the deformed KdV, namely the possibility of accelerated (decelerated) soliton motion.

2.2. Exact soliton solutions

For the deformed KdV equations (1) and (2) no exact solution, except a few particular solutions, could be found [2]. We derive for the same equation exact N-soliton solutions, which is a clear signature of complete integrability of a nonlinear system. It is important to note that the evolution of the basic KdV field $\partial_t u$ is sustained here from two different sources: $\partial_{t_0} u = \partial_t u|_{c(t)=0}$ and $\partial_{t_d} u = c(t)\partial_{\tilde{c}(t)} u$. The first one is generated by the standard dispersive and nonlinear terms in equation (1), while the second one is sourced by the deforming term w_x. Therefore, the deforming function w satisfying the nonholonomic constraint (2) can be determined self-consistently through the KdV field as

$$w(x,t) = c(t) \int u_{\tilde{c}(t)}\, \mathrm{d}x + c(t), \tag{9}$$

where the arbitrary function $c(t)$ acting as a forcing term sitting at the space boundaries $\lim_{x\to\pm\infty} w(x,t) = c(t)$ arises as an integration constant.

Therefore, we can find by applying the ISM the exact soliton solutions for the basic as well as for the perturbing field, interdependent on each other. Recall that in using the ISM through the associated linear problem, the space-Lax operator $U(\lambda)$ describing the scattering of the Jost functions plays the key role. Only at the final stage do we need to fix the time evolution of the solitons through the time dependence of the spectral data, determined in turn by the asymptotic value of the time-Lax operator $V(\lambda)$. Note that since in the case of the deformed KdV equation the space-Lax operator (4) is given by the same operator as in the standard KdV (7), the steps for its ISM follow the same initial path as for the KdV equation [4]. Therefore, referring the readers to the original literature for details, we produce an explicit form of the N-soliton for the KdV field $u(x)$ as an exact solution to the deformed system (1)–(2), or equivalently to the sixth-order KdV (3) as

$$u_N(x) = 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \det A(x), \tag{10}$$

where the matrix function $A(x)$ is expressed through its elements as

$$A_{nm} = \delta_{nm} + \frac{\beta_n}{\kappa_n + \kappa_m}\, \mathrm{e}^{-(\kappa_n+\kappa_m)x}. \tag{11}$$

Here parameters $\kappa_n, n = 1, 2, \ldots, N$ denote the time-independent zeros of the scattering matrix element $a(\lambda = \lambda_n) = 0$, along the imaginary axis: $\lambda_n = \mathrm{i}\kappa_n$ and $\beta_n(t) = b(\lambda = \lambda_n)$ are the time-dependent spectral data to be determined from $V(\lambda) = V_{\mathrm{kdv}}(\lambda) + V_{\mathrm{def}}(\lambda)$, at $x \to \pm\infty$. We note that due to $u \to 0, w \to c(t)$ at $x \to \pm\infty$, the asymptotic value of (8) $V_{\mathrm{kdv}}(\lambda) \to -4\mathrm{i}\lambda^3\sigma^3$ is the usual one, while that of $V_{\mathrm{def}}(\lambda) \to \frac{\mathrm{i}}{2}\lambda^{-1}c(t)\sigma^3$ determines the crucial effect of the deformation. As a result, we obtain

$$\beta_n(t) = \beta_n(0)\, \mathrm{e}^{-(8\kappa_n^3 t - \frac{\tilde{c}(t)}{\kappa_n})}, \qquad \tilde{c}_t = c \tag{12}$$

determining finally the evolution of the soliton through (11).

The exact N-soliton solution for the deforming function $w(x,t)$ induced through (9) by the solution of the basic field (10), therefore can be given as

$$w_N(x,t) = 2c(t)\frac{\partial^2}{\partial x \partial \tilde{c}(t)}(\ln \det A(x,t)) + c(t), \tag{13}$$

where $A(x,t)$ is the same matrix function (11) with its time dependence (12).

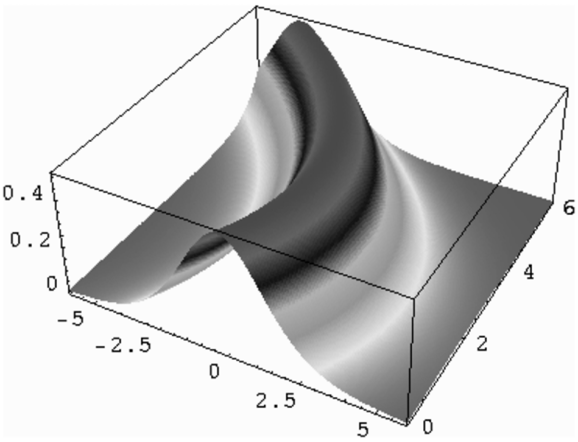

Figure 1. Exact soliton solution $u_1(x,t)$ of the KdV field for the nonholonomically deformed equations (1) and (2) showing an usual localized form of the soliton, but with its unusual decelerating motion, as evident from the bending of soliton trajectory with time.

To examine the deforming effect on solitons in more detail we analyze particular cases of solution (10) and (13) for $N = 1, 2$.

The 1-*soliton* solution of NHD of the KDV equation as reduced from (10) to (13) can be expressed as

$$u_1(x,t) = \frac{v_0}{2}\operatorname{sech}^2\xi, \qquad \xi = \kappa(x+vt)+\phi, \tag{14}$$

$$w_1(x,t) = c(t)(1-\operatorname{sech}^2\xi), \tag{15}$$

with $v = v_0 + v_d$, where $v_0 = 4\kappa^2$ is the usual constant KdV soliton velocity, while $v_d = -\frac{2\tilde{c}(t)}{v_0 t}$ is its unusual time-dependent part induced by the deformation. We stress again that the deforming function is determined by the dynamics of the field u, which in turn is forced self-consistently by the deforming field. Inserting the explicit soliton solutions (14) and (15) for both u and w in the nonholonomic deformation of the KdV (1)–(2) one can directly check the validity of these exact solutions. Note that the time-dependent asymptotic value of the deformation $c(t)$ acts here like a forcing term sitting at the space boundaries, which for $c(t) = c_0 t$ with $c_0 > 0$ forces the soliton to accelerate, while with $c_0 < 0$ makes the soliton decelerate and finally revert its direction (see figure 1). It can also be noted that the original soliton velocity v_0 is less, the deforming velocity v_d is more, which is physically consistent since the forcing term in general must have more of a prominent effect on the slow-moving solitons.

The exact 2-*soliton* in the deformed KdV can be derived similarly from (10) to (13) with $N = 2$ in the explicit form:

$$u_2(x,t) = \frac{2}{D^2}\left(D_{xx}D - D_x^2\right), \qquad D = 1 + \mathrm{e}^{-\xi_1} + \mathrm{e}^{-\xi_2} + p_{12}\,\mathrm{e}^{-(\xi_1+\xi_2)} \tag{16}$$

$$\begin{aligned} w_2(x,t) &= c(t)\left(1 + \frac{2}{D^2}(D\tilde{D}_x - \tilde{D}D_x)\right),\\ \tilde{D} &= \frac{1}{\kappa_1}\mathrm{e}^{-\xi_1} + \frac{1}{\kappa_2}\mathrm{e}^{-\xi_2} + p_{12}\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}\right)\mathrm{e}^{-(\xi_1+\xi_2)}, \end{aligned} \tag{17}$$

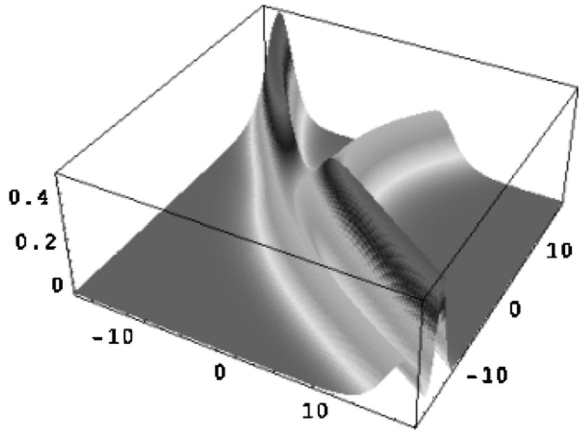

Figure 2. Exact 2-soliton solution $u_2(x,t)$ of the KdV field for the deformed equations (1) and (2) or equivalently for (3). Usual elastic soliton scattering with phase shift is evident, though the dynamics here is dominated by their unusual accelerating motion, reflected in the bending of soliton trajectories.

where the scattering amplitude $p_{12} = \left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2$ and $\xi_n = 2\kappa_n(x + v_n t) + \phi_n, n = 1, 2$ with $v_n = v_{0n} + v_{dn}$. The usual constant soliton velocities $v_{0n} = 4\kappa_n^2, n = 1, 2$ of the KdV equation is boosted here by the time-dependent velocities $v_{dn} = -\frac{2\tilde{c}(t)}{v_{0n}t}, n = 1, 2$, caused by the nonholonomic deformation. The scattering of solitons for the field u as described by the solution (16) with $\tilde{c}(t) = \frac{c_0}{2}t^2$ is depicted in figure 2, which shows the usual elastic collision of solitons as conjectured in [2], but with an unusual dynamics due to the accelerating motion of the solitons.

3. Two-fold integrable hierarchy for the deformed KdV equation

The well-known integrable hierarchy of the standard KdV equation is given by [4] $u_t = B_1\left(\frac{\delta H_{n+1}^{\mathrm{kdv}}}{\delta u}\right) = B_2\left(\frac{\delta H_n^{\mathrm{kdv}}}{\delta u}\right)$, with $B_1 = \frac{\partial}{\partial x}$ and $B_2 = \partial^3 + 2(u\partial + \partial u)$, where $H_n^{\mathrm{kdv}}, n = 1, 2, \ldots$ are the higher Hamiltonians of the KdV hierarchy, e.g., $H_1^{\mathrm{kdv}} = u$, $H_2^{\mathrm{kdv}} = \frac{u^2}{2}$, $H_3^{\mathrm{kdv}} = \frac{1}{3}u^3 - \frac{1}{2}u_x^2$ etc, with $n = 2$ yielding the KdV equation. As has been shown in [3] an integrable hierarchy with the same Hamiltonians exists also for the deformed KdV equation with NHD of the equations as

$$u_t = B_1\left(\frac{\delta H_{n+1}^{\mathrm{kdv}}}{\delta u} - w\right), \qquad B_2(w) = 0. \tag{18}$$

For $n = 2$ one obviously recovers the known deformed KdV equations (1) and (2). This usual type of hierarchy with higher dispersions can be generated from the AKNS Lax pair, where the space-Lax operator remains same as the original one (7), but the time-Lax operator is changed to $V(\lambda) = V_{\mathrm{kdv}}^{(n)}(\lambda) + V_{\mathrm{def}}(\lambda)$. Here the deforming part $V_{\mathrm{def}}(\lambda)$ is the same as (5)–(6), while $V_{\mathrm{kdv}}^{(n)}(\lambda)$ is the higher generalization of (8), where a polynomial in the spectral parameter λ up to nth power appears. Such higher order time-Lax operator can be constructed by expanding this matrix in the powers of λ and determining the matrix coefficients recursively from the flatness condition, as done in the standard AKNS treatment [4]. Another simpler solution for this problem based on the dimensional analysis and identification of the building blocks of the Lax operators has been proposed recently [5].

We discover, apart from the usual integrable hierarchy given above, an unusual hierarchy for the nonholonomic deformation of the KdV equation, which can be represented by the same deformed KdV (1) but with higher order nonholonomic constraints on the deforming function w. This novel integrable hierarchy can be generated as the flatness condition of a Lax pair, where the space-Lax operator remains as (7), while in the time-Lax operator only the deforming part changes as $V(\lambda) = V_{\mathrm{kdv}}(\lambda) + V_{\mathrm{def}}^{(n)}(\lambda)$. Unlike the above KdV hierarchy, $V_{\mathrm{def}}^{(n)}(\lambda) = \frac{1}{2}\sum_j^n \lambda^{-j} G^{(j)}$, $j = 1, 2, \ldots n$ contains only negative powers of λ up to n, with n number of deforming matrix coefficients $G^{(j)}$. The consistency condition generates this new integrable hierarchy of nonholonomic deformations given by the same deformed evolution equation (1), but where constraint (2) is generalized now to nth order through a set of coupled differential equations:

$$
\begin{aligned}
&u_t - u_{xxx} - 6uu_x = G_{12}^{(1)},\\
&G_x^{(1)} = \mathrm{i}[U^{(0)}, G^{(1)}] + \mathrm{i}[\sigma_3, G^{(2)}],\\
&\ldots \qquad \ldots,\\
&G_x^{(n-1)} = \mathrm{i}[U^{(0)}, G^{(n-1)}] + \mathrm{i}[\sigma_3, G^{(n)}],\\
&G_x^{(n)} = \mathrm{i}[U^{(0)}, G^{(n)}].
\end{aligned}
\tag{19}
$$

Clearly this hierarchy reduces to NHD of the KdV (1)–(2) for $n = 2$, with the deforming operator $V_{\mathrm{def}}^{(2)}(\lambda)$ reducing to (5)–(6).

4. Concluding remarks

We prove here a number of conjectures on the recently proposed nonholonomic deformation of the KdV equation, unraveling its several unexpected features. In particular, we construct an AKNS-type matrix Lax pair for this deformed KdV equation, showing an intriguing connection between the deformation of the time evolution in the associated linear problem and the deformation of the nonlinear equation. Applying the inverse scattering method we find exact N-soliton solutions for the basic as well as the deforming field for the deformed KdV equation, which also gives the solution of the sixth-order KdV equation. Such solitons exhibit in spite of the isospectral flow an unusual accelerated motion, which is however consistent with the particle motion under force. The deforming function $w(x, t)$, as seen from (18), enters in the original hierarchy of the KdV equation as a perturbation together with a nonholonomic differential constraint on it. The driving term sitting at the space boundaries $c(t) = w(\pm\infty, t)$, which can be an arbitrary function in time, forces the field soliton to accelerate or decelerate, with the perturbing soliton itself created by the field soliton in a self-consistent way.

We discover also an unique two-fold integrable hierarchy for this deformed system, one with the usual higher dispersion found already and the other with new increasingly higher order nonholonomic deformation. Extension of nonholonomic deformation to other integrable models such as NLS, sine-Gordon, mKdV, etc is under investigation [5].

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